External Littelmann Paths of Kashiwara Crystals of Type A rank $e$

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Let $\mathcal{G}$ over $\mathbb{C}$ be an affine Lie algebra of type $A$ rank $e$.

Let $\Lambda$ be a dominant integral weight.

$V(\Lambda)$ is the highest weight representation.

$\alpha_0, \ldots, \alpha_{e-1}$ are a simple roots.

Let $Q$ be the $\mathbb{Z}$-lattice generated by the simple roots.

$P(\Lambda)$ is the set of weights of the weight spaces of $V(\Lambda)$, which are of the form $\lambda = \Lambda - \sum_{i=0}^{e-1} k_i \alpha_i$, where $(k_0, \ldots, k_{e-1})$ called the content for the weight.
There are two important ways of describing $B(\Lambda)$:

- By $e$-regular multipartitions, sets of $r$ partitions.
- By a geometric way called Littelmann paths.

**The problem**: We get these two ways recursively, so maybe there is a way to pass from a way to another, such that we use one way to determine the other way directly?
The Littelmann path model for $B(\Lambda)$

- An $LS$-path $\pi(t)$ is a piecewise linear path in the weight space of the Lie algebra $G$,

$$\mathfrak{h}^* = \langle \Lambda_0, \Lambda_1 \ldots, \Lambda_{e-1}, \delta \rangle$$

and parameterized by the real interval $[0, 1]$, with $\pi(0) = 0$.

- The set of paths obtained by acting with various $f_i$, starting with the path from 0 to $\Lambda$ has the structure of a Kashiwara crystal $B(\Lambda)$, as proven by Littelmann [L].

- The straight paths in the piecewise linear paths are rational multiples of weight vectors of defect zero.

- The endpoints of these straight paths are called the corner points, and the final corner point $\pi(1) \in P(\Lambda)$ is the weight of the basis element in $V(\Lambda)$. 
The Littelmann path model for $B(\Lambda)$

Definition

A Littelmann path $\pi(t)$ has an $LS$-representation if there is a sequence of defect 0 weights $\nu_p, \ldots, \nu_0$ and rational numbers $a_{p+1} = 0, a_p, \ldots, a_0 = 1$ such that for $t \in [a_{i+1}, a_i]$, we have

$$\pi(t) = \pi(a_{i+1}) + (t - a_{i+1})\nu_i$$
The Littelmann path model for $B(\Lambda)$

- We define a function $H^\pi_\epsilon(t) = \langle \pi(t), h_\epsilon \rangle$, which is simply the projection of the path onto the coefficient of $\Lambda_\epsilon$. We then set

$$m_\epsilon = \min_t (H^\pi_\epsilon(t)).$$

This minimum is always achieved at one of the finite set of corner weights.

- We let $\mathcal{P}_{\text{int}}$ be the set of paths for which this $m_\epsilon$ is an integer for all $\epsilon \in I$. 

The Littelmann path model for $B(\Lambda)$

**Definition**

Littelmann’s function $f_\epsilon$ is given on $\mathcal{P}_{int}$ as follows:

- If $H_\epsilon^\pi(1) = m_\epsilon$, then $f_\epsilon(\pi) = 0$
- Set

$$t_0 = \max_t \{ t \in [0, 1] \mid H_\epsilon^\pi(t) = m_\epsilon \}$$

$$t_1 = \min_t \{ t \in [t_0, 1] \mid H_\epsilon^\pi(t) = m_\epsilon + 1 \}$$

then

$$f_\epsilon(\pi)(t) = \begin{cases} 
\pi(t) & t \in [0, t_0] \\
\pi(t_0) + s_\epsilon(\pi(t) - \pi(t_0)) & t \in [t_0, t_1] \\
\pi(t) - \alpha_\epsilon & t \in [t_1, 1]
\end{cases}$$
Example $\Lambda = \Lambda_0 + \Lambda_1, e = 2$
Let

\[ \Lambda = \Lambda_{k_1} + \Lambda_{k_2} + \cdots + \Lambda_{k_r} \]

\[ = c_0 \Lambda_0 + \cdots + c_{e-1} \Lambda_{e-1} \]

be an integral dominant weight. We define the residue for the node \((i,j)\) in Young diagram that corresponds to e-regular multipartition \(\lambda\) is

\[ k_\ell + j - i \]

This will be called a \(k_\ell\)-corner partition.
We describe the Littelmann path by projecting the weight space onto subspace generated by the fundamental weights.

And we concentrated with the external vertices in $B(\Lambda)$. 
A Littelmann path will be called \textit{standard} if the rational numbers are of the form

\[ e_m = \frac{c_m}{d_m}, \]

where \( d_m \) was the number of nodes added to a defect 0 multipartition with first row \( m - 1 \) to get that for \( m \), and \( c_m \) is the number of nodes in the intersection of these added nodes with our multipartition.
The case of $\Lambda = \Lambda_0, e = 2, r = 1$

**Definition**

Let $\lambda$ be an $e-$regular partition, for $e = 2, r = 1$, we call $\lambda$ alternating if the parity of the rows alternates between odd and even.

**Definition**

Segment is a sequence of rows differing by one.

**Definition**

$\theta_n$ is the hub of defect 0 partition that is a triangle partition, and it equal to

$$
\theta_n = \begin{cases} 
[-n, n+1] & n \equiv 1 \ mod(2) \\
[n+1, -n] & n \equiv 0 \ mod(2)
\end{cases}
$$
The case of $\Lambda = \Lambda_0$, $e = 2$, $r = 1$

**Theorem**

Let $e = 2$ and let $\lambda$ be an alternating partition from an external vertex of the reduced crystal for $\Lambda_0$. Let $b_i$ be the number of rows down to the bottom of segment $i$, and let $n'_i$ be the first row of segment $i$ extended upward in a stairstep. Then the LS-representation of the Littelmann path is

$$\left(\theta_{n_1}, \theta_{n_1-1}, \ldots, \theta_{b_r}; \frac{b_1}{n'_1}, \frac{b_1}{n'_1 - 1}, \ldots, \frac{b_1}{n'_2 + 1}, \frac{b_2}{n'_2}, \frac{b_2}{n'_2 - 1}, \ldots, \frac{b_2}{n'_3 + 1}, \frac{b_3}{n'_3}, \ldots, \frac{b_r}{b_r}\right)$$
Example

\[ \lambda = (8, 7, 4, 1). \]
There are three segments:

\[ \mu_1 = (8, 7), \mu_2 = (4), \mu_3 = (1). \]

Then

\[ n_1 = 8, b_0 = 0, n'_1 = 8 \]
\[ n_2 = 4, b_1 = 2, n'_2 = 6 \]
\[ n_3 = 1, b_2 = 3, n'_3 = 4 \]
\[ n_4 = 0, b_3 = 4, n'_4 = 4. \]

so the LS-representation of the Littelmann path is

\[ (\theta_8, \theta_7, \theta_6, \theta_5, \theta_4 : \frac{2}{8}, \frac{2}{7}, \frac{3}{6}, \frac{3}{5}, \frac{4}{4}) \]
Residue-homogeneous multipartitions for $e = 2$

The following condition will ensure that the end points of all the rows would have the same residue 0 or 1.

Definition

A multipartition will be called *residue homogeneous* if it satisfies the following conditions:

- each partition has rows of alternating parity,
- all zero corner partitions have first rows of the same parity and the 1-corner of opposite parity,

Definition

A residue-homogeneous multipartition will be called *strongly residue homogeneous* if it satisfies the following conditions:

- For every non-initial partition, if $n$ is the length of the first row, then the previous partition ends in a triangle with $n$ rows and columns.
Definition

In the case $e = 2$, $r > 1$, a segment is a sequence of rows with difference 1, the segment can go on the next partition if the first row of the next partition is equal the number $l$ of the column of the previous partition, and if we are in the boundary between 0 and 1, the length of the first row should be $l \pm 1$.

We succeeded in connecting between the multipartition and the Littelmann path by cutting the Young diagram to segments.
Example for the connections with the multipartition model for $\Lambda = a\Lambda_0 + b\Lambda_1$, $e = 2$, $r = a + b$
Example: \( \Lambda = 2\Lambda_0 + \Lambda_1, \; e = 2, \lambda = [(5, 2, 1), (1), \phi] \)

We get this external multipartition by \((f_0 f_1 f_0 f_1^2 f_0^4) u_\phi\). We have three segments,

\[
\begin{align*}
c_1 &= 2, \; c_2 = 2, \; c_3 = 3, \; c_4 = 1, \; c_5 = 1 \\
d_1 &= 2, \; d_2 = 5, \; d_3 = 8, \; d_4 = 11, \; d_5 = 14
\end{align*}
\]

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 \\
1 & 0 \\
0 \\
0 \\
\phi
\end{array}
\]
Example: $\Lambda = 2\Lambda_0 + \Lambda_1$, $e = 2$,
$\lambda = [(5, 2, 1), (1), (\phi)]$

The LS-representation is:

$$(\psi_5^-, \psi_4^-, \psi_3^-, \psi_2^-, \psi_1^- : \frac{1}{14}, \frac{1}{11}, \frac{3}{8}, \frac{2}{5}, \frac{2}{2})$$

$\psi_m^-$ is the weight of the multipartition of defect 0, with Weyl group word beginning in $s_0$, where all the 0-corner partitions have the same length of first row $m$.
We write suitable equations to calculate $c_m$ and $d_m$ directly from the multipartition[OS1]
The graphic diagram reflects the decomposition into segments of the multipartition. If the vertex of $P(\Lambda)$ is external, then the graph lies in the second or fourth quadrant, with a long straight path of length giving the rows of the segment, and an oscillating part whose length depends on the offset between the segments.
Reduced crystal, $e = 2, \Lambda = 2\Lambda_0 + \Lambda_1$, with multipartitions
The Main Result

Theorem

In the case $e = 2$, the Littelmann path corresponding to a strongly residue homogeneous multipartition is standard.

The set of all strongly residue-homogeneous multipartitions can be determined non-recursively, and then the corresponding Littelmann path constructed.
The general case is too complicated for finding formulas, but for periodic Weyl group elements it is possible. We consider periods 3 and 4.

- Periodic Weyl group element of period 3.
  We choose to add nodes on our multipartitions in the order $0, 1, 2, 0, 1, 2, \ldots$.

- Periodic Weyl group element of period 4.
  The reduced word are those of the form

$$\left(s_1 s_0 s_2 s_0\right)\left(s_1 s_0 s_2 s_0\right) \cdots = s_1 s_2 s_0 s_2 s_1 s_2 s_0 s_2 \cdots$$
\[ \Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2, \ e = 3, \ \lambda = [(15, 4, 2), \phi, \phi] \]

For 3 period Weyl words we get the the long path and oscillating paths corresponding to segments like the case \( e = 2 \)


Thank you for your attention.
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THE END