External Littelmann Paths of Kashiwara Crystals of Type A rank e

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Combinatorics and Representation Theory
July 23-27, 2018

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Kashiwara Crystals $B(\Lambda)$

- Let $\mathcal{G}$ over $\mathbb{C}$ be an affine Lie algebra of type A rank e.
- Let Λ be a dominant integral weight.
- $V(\Lambda)$ is the highest weight representation.
- $\alpha_0, \ldots, \alpha_{e-1}$ are a simple roots.
- Let Q be the $\mathbb{Z}$-lattice generated by the simple roots.
- $P(\Lambda)$ is the set of weights of the weight spaces of $V(\Lambda)$, which are of the form $\lambda = \Lambda - \sum_{i=0}^{e-1} k_i\alpha_i$, where $(k_0, \ldots, k_{e-1})$ called the content for the weight.

Models of the Kashiwara Crystal of type A

There are two important ways of describing $B(\Lambda)$:

- By e-regular multipartions, sets of r partitions.
- By a geometric way called Littelmann paths.

The problem: We get these two ways recursively, so maybe there is a way to pass from a way to another, such that we use one way to determine the other way directly ?

The Littelmann path model for $B(\Lambda)$

- An LS-path $\pi(t)$ is a piecewise linear path in the weight space of the Lie algebra $\mathcal{G}$,

$$\mathfrak{h}^* =< \Lambda_0, \Lambda_1 \ldots, \Lambda_{e-1}, \delta >$$

  and parameterized by the real interval $[0, 1]$, with $\pi(0) = 0$.
- The set of paths obtained by acting with various f_i, starting with the path from 0 to Λ has the structure of a Kashiwara crystal $B(\Lambda)$, as proven by Littelmann [L].
- The straight paths in the piecewise linear paths are rational multiples of weight vectors of defect zero.
- The endpoints of these straight paths are called the corner points, and the final corner point $\pi(1) \in P(\Lambda)$ is the weight of the basis element in $V(\Lambda)$.

The Littelmann path model for $B(\Lambda)$

Definition

A Littelmann path $\pi(t)$ has an LS-representation if there is a sequence of defect 0 weights $\nu_p, \ldots, \nu_0$ and rational numbers $a_{p+1} = 0, a_p, \ldots, a_0 = 1$ such that for $t \in [a_{i+1}, a_i]$, we have

$$\pi(t) = \pi(a_{i+1}) + (t - a_{i+1})\nu_i$$

The Littelmann path model for $B(\Lambda)$

- We define a function $H_\epsilon^\pi(t) =< \pi(t), h_\epsilon >$, which is simply the projection of the path onto the coefficient of Λ_ϵ. We then set

$$m_\epsilon = \min_t(H_\epsilon^\pi(t)).$$

  This minimum is always acheived at one of the finite set of corner weights.
- We let $\mathcal{P}_{int}$ be the set of paths for which this m_ϵ is an integer for all $\epsilon \in I$.

The Littelmann path model for $B(\Lambda)$

Definition

Littelmann's function f_ϵ is given on $\mathcal{P}_{int}$ as follows:

- If $H_\epsilon^\pi(1) = m_\epsilon$, then $f_\epsilon(\pi) = 0$
- Set

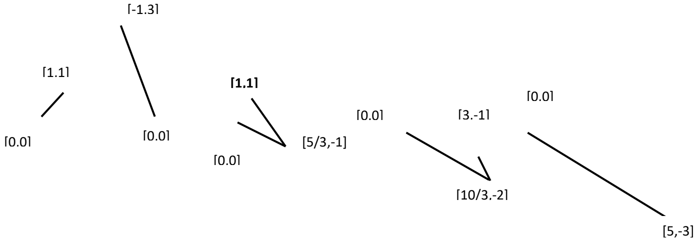
$$t_0 = \max_t\{t \in [0,1] \mid H_\epsilon^\pi(t) = m_\epsilon\}$$

$$t_1 = \min_t\{t \in [t_0,1] \mid H_\epsilon^\pi(t) = m_\epsilon + 1\}$$

then

$$f_\epsilon(\pi)(t) = \begin{cases} \pi(t) & t \in [0,t_0] \\ \pi(t_0) + s_\epsilon(\pi(t) - \pi(t_0)) & t \in [t_0,t_1] \\ \pi(t) - \alpha_\epsilon & t \in [t_1,1] \end{cases}$$

Example $\Lambda = \Lambda_0 + \Lambda_1, e = 2$



Multipartitions

Let

$$\Lambda = \Lambda_{k_1} + \Lambda_{k_2} + \cdots + \Lambda_{k_r}$$
$$= c_0\Lambda_0 + \cdots + c_{e-1}\Lambda_{e-1}$$

be an integral dominant weight. We define the residue for the node (i, j) in Young diagram that corresponds to e-regular multipartition λ is

$$k_\ell + j - i$$

This will be called a k_ℓ-corner partition.

Graphic Diagram of Littelmann Paths

- We describe the Littelmann path by projecting the weight space onto subspace generated by the fundamental weights.
- And we concentrated with the external vertices in $B(\Lambda)$.

Standard Littelmann paths for $e = 2$

A Littelmann path will be called *standard* if the rational numbers are of the form

$$e_m = \frac{c_m}{d_m}, \tag{1}$$

where d_m was the number of nodes added to a defect 0 multipartion with first row $m - 1$ to get that for m, and c_m is the number of nodes in the intersection of these added nodes with our multipartition .

The case of $\Lambda = \Lambda_0, e = 2, r = 1$

Definition

Let λ be an $e-$regular partition, for $e = 2, r = 1$, we call λ alternating if the parity of the rows alternates between odd and even.

Definition

Segment is a sequence of rows differing by one.

Definition

θ_n is the hub of defect 0 partition that is a triangle partition, and it equal to

$$\theta_n = \begin{cases} [-n, n+1] & n \equiv 1 \ mod(2) \\ [n+1, -n] & n \equiv 0 \ mod(2) \end{cases}$$

The case of $\Lambda = \Lambda_0, e = 2, r = 1$

Theorem

Let $e = 2$ and let λ be an alternating partition from an external vertex of the reduced crystal for Λ_0. Let b_i be the number of rows down to the bottom of segment i, and let n'_i be the first row of segment i extended upward in a stairstep. Then the LS-representation of the Littelmann path is

$$(\theta_{n_1}, \theta_{n_1-1}, \dots, \theta_{b_r};$$

$$\frac{b_1}{n'_1}, \frac{b_1}{n'_1-1}, \dots, \frac{b_1}{n'_2+1}, \frac{b_2}{n'_2}, \frac{b_2}{n'_2-1}, \dots, \frac{b_2}{n'_3+1}, \frac{b_3}{n'_3}, \dots, \frac{b_r}{b_r})$$

The case of $e = 2$, $\Lambda = \Lambda_0$

Example

$\lambda = (8, 7, 4, 1)$.

There are three segments :

$$\mu_1 = (8, 7), \mu_2 = (4), \mu_3 = (1).$$

Then

$n_1 = 8, b_0 = 0, n'_1 = 8$

$n_2 = 4, b_1 = 2, n'_2 = 6$

$n_3 = 1, b_2 = 3, n'_3 = 4$

$n_4 = 0, b_3 = 4, n'_4 = 4.$

so the the LS-representation of the Littelmann path is

$$(\theta_8, \theta_7, \theta_6, \theta_5, \theta_4 : \frac{2}{8}, \frac{2}{7}, \frac{3}{6}, \frac{3}{5}, \frac{4}{4})$$

Residue-homogeneous multipartitions for $e = 2$

The following condition will ensure that the end points of all the rows would have the same residue 0 or 1.

Definition

A multipartition will be called *residue homogeneous* if it satisfies the following conditions:

- each partition has rows of alternating parity,
- all zero corner partitions have first rows of the same parity and the 1-corner of opposite parity,

Definition

A residue-homogeneous multipartition will be called *strongly* residue homogeneous if it satisfies the following conditions:

- For every non-initial partition, if n is the length of the first row, then the previous partition ends in a triangle with n

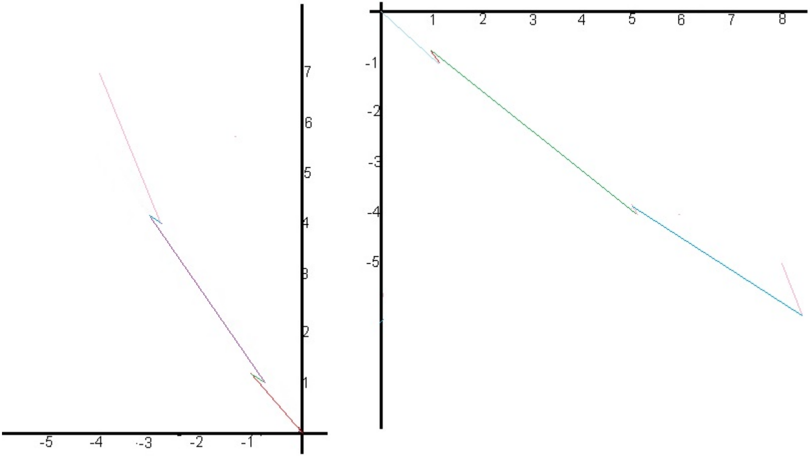
Residue-homogeneous multipartition for $\Lambda = a\Lambda_0 + b\Lambda_1, e = 2, r = a + b$

Definition

In the case $e = 2, r > 1$, a segment is a sequence of rows with difference 1, the segment can go on the next partition if the first row of the next partition is equal the number l of the column of the previous partition, and if we are in the boundary between 0 and 1, the length of the first row should be $l \pm 1$.

We succeeded in connecting between the multipartition and the Littelmann path by cutting the Young diagram to segments.

Example for the connections with the multipartition model for $\Lambda = a\Lambda_0 + b\Lambda_1, e = 2, r = a + b$



$[(5, 2, 1), (1), \emptyset]; [(6, 3, 2, 1), (2), \emptyset]$

Example :$\Lambda = 2\Lambda_0 + \Lambda_1$, $e = 2$, $\lambda = [(5, 2, 1), (1), \phi]$

We get this external multipartition by $(f_0 f_1 f_0 f_1^2 f_0^4) u_\phi$. W e have three segments,

$c_1 = 2, c_2 = 2, c_3 = 3, c_4 = 1, c_5 = 1$

$d_1 = 2, d_2 = 5, d_3 = 8, d_4 = 11, d_5 = 14$

0	1	0	1	0
1	0			
0				

0

ϕ

Example :$\Lambda = 2\Lambda_0 + \Lambda_1$, $e = 2$, $\lambda = [(5,2,1),(1),\phi]$

The LS-representation is :

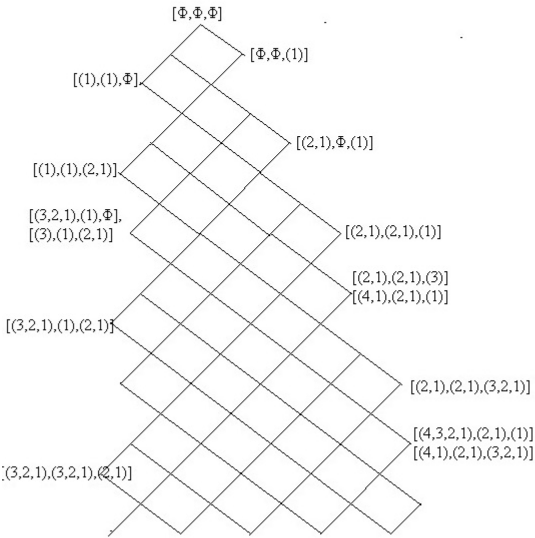
$$(\psi_5^-, \psi_4^-, \psi_3^-, \psi_2^-, \psi_1^- : \frac{1}{14}, \frac{1}{11}, \frac{3}{8}, \frac{2}{5}, \frac{2}{2})$$

ψ_m^- is the weight of the multipartition of defect 0, with Weyl group word beginning in s_0, where all the 0-corner partitions have the same length of first row m.
We write suitable equations to calculate c_m and d_m directly from the multipartition[OS1]

External Littelmann Paths

The graphic diagram reflects the decomposition into segments of the multipartition. If the vertex of $P(\Lambda)$ is external, then the graph lies in the second or fourth quadrant, with a long straight path of length giving the rows of the segment, and an oscillating part whose length depends on the offset between the segments.

Reduced crystal, $e = 2, \Lambda = 2\Lambda_0 + \Lambda_1$, with multipartitions



The Main Result

Theorem

In the case $e = 2$, the Littelmann path corresponding to a strongly residue homogeneous multipartition is standard

The set of all strongly residue-homogeneous multipartitions can be determined non-recursively, and then the corresponding Littelmann path constructed.

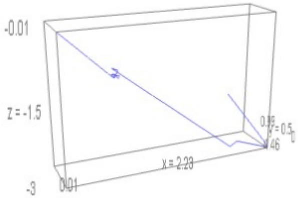
Standard Littelmann paths for $\Lambda = a\Lambda_0 + b\Lambda_1 + c\Lambda_2, e = 3$

The general case is too complicated for finding formulas, but for periodic Weyl group elements it is possible. We consider periods 3 and 4.

- Periodic Weyl group element of period 3.
  We choose to add nodes on our multipartitions in the order $0, 1, 2, 0, 1, 2 \ldots$.
- Periodic Weyl group element of period 4.
  The reduced word are those of the form

$$(s_1 s_0 s_2 s_0)(s_1 s_0 s_2 s_0) \cdots = s_1 s_2 s_0 s_2 s_1 s_2 s_0 s_2 \ldots$$

$\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2, e = 3, \lambda = [(15, 4, 2), \phi, \phi]$



For 3 period Weyl words we get the the long path and oscillating paths corresponding to segments like the case $e = 2$

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Thank you for your attention.

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